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Heisenberg equations of motion for the spin-$\frac{3}{2}$ field in the presence of an interaction

A K Nagpal
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India

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Abstract. The Rarita–Schwinger spin-$\frac{3}{2}$ field interacting with a Dirac field and a scalar field (external) is found to satisfy the Heisenberg equations of motion, in the weak-field limit. This is analogous to the result, for the case of spin-$\frac{3}{2}$ field minimally coupled with external electromagnetic field, recently obtained by Mainland and Sudarshan.

1. Introduction

A few years ago Gupta and Repko (1969) claimed that for the case of a spin-$\frac{3}{2}$ Rarita–Schwinger field (Rarita and Schwinger 1941) minimally coupled with a non-external (or quantised) electromagnetic field, the canonical variables of the type suggested by Johnson and Sudarshan (1961) do not satisfy the Heisenberg equations of motion. They obtained a modified set of canonical variables for the interacting system, up to $O(e^2)$, which are consistent with the Heisenberg equations of motion. The same set of canonical variables was constructed by Kimel and Nath (1972) for the above interaction using the Yang and Feldman (1950) approach, to the same order e^2. The straightforward extension of the analysis of Kimel and Nath does not give the correct modified canonical variable to higher order in coupling constant, say $O(e^4)$. But we have obtained these (Nagpal 1977) by some manipulations while establishing the complete equivalence of results of: (i) the Yang–Feldman technique; and (ii) the canonical or Schwinger action principle approach (Schwinger 1953).

Similarly for a non-electromagnetic interaction, e.g. the interaction of the spin-$\frac{3}{2}$ field with a Dirac field and a scalar field, in the chiral invariant manner proposed by Nath *et al* (1971), the modified canonical variables were obtained by Nath *et al* (1972 *Florida State University Preprint* HEP 72-8-4, unpublished), again up to order g^2, g being the major coupling constant. Here the extension to higher order, say up to $O(g^4)$, can easily be carried out (Nagpal 1977).

On the other hand Mainland and Sudarshan (1973) had shown that for the case of a spin-$\frac{3}{2}$ field coupled to the *external* electromagnetic field, the Heisenberg equations of motion can be satisfied provided the explicit space–time dependence of the electromagnetic field is considered correctly. Then due to the presence of the constraint relations (Johnson and Sudarshan 1961), the spin-$\frac{3}{2}$ field functions also acquire explicit space–time dependence from the inherent explicit space–time dependence in the external electromagnetic field. Therefore the Heisenberg equation of motion takes the form

$$d^{\nu}\psi = -i[\psi(x), P^{\nu}] + \partial^{\nu}\psi \tag{1.1}$$

where d^ν and ∂^ν denote the full and partial derivative respectively on the field ψ, while P^ν is the energy–momentum vector for the interacting system. This form (1.1) is consistent with the remark of Johnson and Sudarshan (1961) that the kinematics of the spin-$\frac{3}{2}$ field necessarily involve the dynamics.

In this paper we find that the Heisenberg equations of motion (1.1) can be shown to be satisfied for the spin-$\frac{3}{2}$ field coupled to a Dirac field and an external scalar field. For the present case the Lagrangian and Heisenberg equations of motion are found to be identical, a result analogous to the result by Mainland and Sudarshan (1973) for the case of a spin-$\frac{3}{2}$ field minimally coupled to the external electromagnetic field.

2. (Anti-)Commutation relations for the interacting spin-$\frac{3}{2}$ field system

The Lagrangian density expression for the spin-$\frac{3}{2}$ field system in interaction with a Dirac field and an external scalar field, is (Nath *et al* 1971)

$$\begin{aligned}\mathscr{L} &= \mathscr{L}^{\mathrm{RS}} + \mathscr{L}^{\mathrm{Dirac}} + \mathscr{L}^{\mathrm{int}} \\ &= -\bar{\psi}^\mu[(-\mathrm{i}\gamma\,.\,\mathrm{d}+M)\psi_\mu + \mathrm{i}(\gamma_\mu \mathrm{d}\,.\,\psi + \mathrm{d}_\mu\gamma\,.\,\psi) + \gamma_\mu(\mathrm{i}\gamma\,.\,\mathrm{d}+M)\gamma\,.\,\psi] \\ &\quad - \bar{\psi}(-\mathrm{i}\gamma\,.\,\mathrm{d}+m)\psi + g[\bar{\psi}^\mu(g_{\mu\nu}+\gamma_\mu\gamma_\nu)\psi + \bar{\psi}(g_{\nu\mu}+\gamma_\nu\gamma_\mu)\psi^\mu]\,\partial^\nu\phi \end{aligned} \tag{2.1}$$

where ψ^μ and ψ represent the spin-$\frac{3}{2}$ field and the Dirac field with masses M and m respectively, ϕ being the external scalar field, $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, $\bar{\psi}^\mu = \psi^{+\mu}\gamma^0$ etc as usual. The equations of motion and the constraint relations derived from (2.1) are

$$(-\mathrm{i}\gamma\,.\,\mathrm{d}+M)\psi_\mu + \mathrm{i}(\gamma_\mu\, \mathrm{d}\,.\,\psi + \mathrm{d}_\mu\gamma\,.\,\psi) + \gamma_\mu(\mathrm{i}\gamma\,.\,\mathrm{d}+M)(\gamma\,.\,\psi) - g(g_{\mu\nu}+\gamma_\mu\gamma_\nu)\psi\,\partial^\nu\phi = 0 \tag{2.2}$$

$$(-\mathrm{i}\gamma\,.\,\mathrm{d}+m)\psi - g(g_{\nu\mu}+\gamma_\nu\gamma_\mu)\psi^\mu\,\partial^\nu\phi = 0 \tag{2.3}$$

$$(\tfrac{2}{3}\mathrm{i}\boldsymbol{\gamma}\,.\,\mathbf{d}+M)(\boldsymbol{\gamma}\,.\,\boldsymbol{\psi}) = -\mathrm{i}\,\mathbf{d}\,.\,\boldsymbol{\psi}^{3/2} + g(\boldsymbol{\gamma}\,.\,\boldsymbol{\partial}\phi)\psi \tag{2.4}$$

and

$$(\gamma\,.\,\psi) = \frac{g}{M}\gamma_\mu\psi\,\partial^\mu\phi - \frac{2g\mathrm{i}}{3M^2}\mathrm{d}^\mu[(g_{\mu\nu}+\gamma_\mu\gamma_\nu)\psi\,\partial^\nu\phi] \tag{2.5}$$

where

$$\psi_k^{3/2}(x) = \psi_k(x) + \tfrac{1}{3}\gamma_k\boldsymbol{\gamma}\,.\,\boldsymbol{\psi}(x) \equiv P_{kl}\psi_l(x).$$

The relation (2.5) gives ψ^0 in terms of ψ_k and ψ etc, ψ_k being again not all independent but rather related by the constraints (2.4). Defining the generator G, corresponding to the infinitesimal field variations, namely

$$G = \tfrac{1}{2}\int \mathrm{d}^3x\{\mathrm{i}[\psi_k^{+3/2}\delta\psi_k + \tfrac{2}{3}(\boldsymbol{\psi}^+\,.\,\boldsymbol{\gamma})\delta(\boldsymbol{\gamma}\,.\,\boldsymbol{\psi}) + \psi^+\delta\psi] + \mathrm{HC}\} \tag{2.6}$$

again the variations $\delta\psi_k^{3/2}$, $\delta(\boldsymbol{\gamma}\,.\,\boldsymbol{\psi})$ and $\delta\psi$ are not independent but are related by

$$-(\tfrac{2}{3}\mathrm{i}\boldsymbol{\gamma}\,.\,\mathbf{d}+M)\delta(\boldsymbol{\gamma}\,.\,\boldsymbol{\psi}) - \mathrm{i}\,\mathbf{d}\,.\,\delta\boldsymbol{\psi}^{3/2} + g(\boldsymbol{\gamma}\,.\,\boldsymbol{\partial}\phi)\delta\psi = 0. \tag{2.7}$$

Using Schwinger's action principle (Schwinger 1953), the (anti-)commutation relations can be obtained. This has been done by Hagen (1971). We list below the required results, for ϕ external

$$[\psi_k^{3/2}(x), \psi_l^{+3/2}(x')]_+ = P_{kn}\{g_{nm} - \tfrac{2}{3}\,d_n[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}\,d_m\}P_{ml}\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$[\psi_k^{3/2}(x), \psi^+(x')]_+ = -\tfrac{2}{3}igP_{kl}\,d_l[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$[\psi(x), \psi^+(x')]_+ = M^2[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$[\psi(x), \psi_k^{+3/2}(x')]_+ = \tfrac{2}{3}ig(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}\,d_lP_{lk}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{2.8}$$

and so on.

3. Heisenberg equations of motion in the presence of an interaction

The energy–momentum vector P^ν for the present interacting system is given by

$$P^\nu = \int\left(\frac{\partial\mathcal{L}}{\partial\partial_0\psi_\mu}\,d^\nu\psi_\mu + \frac{\partial\mathcal{L}}{\partial d_0\psi}\,d^\nu\psi - g^{\nu0}\mathcal{L}\right)d^3x$$
$$= i\int[\psi_k^{+3/2}\,d^\nu\psi_k^{3/2} + \tfrac{2}{3}(\boldsymbol{\psi}^+\cdot\boldsymbol{\gamma})\,d^\nu(\boldsymbol{\gamma}\cdot\boldsymbol{\psi}) + \psi^+\,d^\nu\psi]\,d^3x \tag{3.1}$$

where $d^0\psi_k^{3/2}$ is given by

$$d^0\psi_k^{3/2} = -i\gamma_0[P_{kl}(-i\boldsymbol{\gamma}\cdot\mathbf{d} + M)\psi_l + iP_{kl}\,d_l(\gamma\cdot\psi) - gP_{kl}\psi\,\partial^l\phi]. \tag{3.2}$$

Using the primary constraint (2.4), we can calculate

$$d^\nu(\boldsymbol{\gamma}\cdot\boldsymbol{\psi}) \equiv d^\nu\chi = (\tfrac{2}{3}i\boldsymbol{\gamma}\cdot\mathbf{d} + M)^{-1}[-i\,\mathbf{d}\cdot d^\nu\boldsymbol{\psi}^{3/2} + g(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)d^\nu\psi + g(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi]. \tag{3.3}$$

Substituting for this and $\chi^+ \equiv (\boldsymbol{\gamma}\cdot\boldsymbol{\psi})^+$, P^ν takes the form

$$\begin{aligned}P^\nu = i\int d^3x\,\{&\psi_k^{+3/2}[\delta_{kl} + \tfrac{2}{3}\,d_k(M^2 - \tfrac{4}{9}\mathbf{d}^2)^{-1}\,d_l]\,d^\nu\psi_l^{3/2}\\ &-\tfrac{2}{3}ig\psi^+(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)(M^2 - \tfrac{4}{9}\mathbf{d}^2)^{-1}\,d_l\,d^\nu\psi_l^{3/2}\\ &+\tfrac{2}{3}g[i\boldsymbol{\psi}^{+3/2}\cdot\mathbf{d} + g\psi^+(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)](M^2 - \tfrac{4}{9}\mathbf{d}^2)^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)d^\nu\psi\\ &+\tfrac{2}{3}g[i\boldsymbol{\psi}^{+3/2}\cdot\mathbf{d} + g\psi^+(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)](M^2 - \tfrac{4}{9}\mathbf{d}^2)^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi\}.\end{aligned} \tag{3.4}$$

Now, evaluating $-i[\psi(x), P^\nu]$, which comes out to be

$$-i[\psi(x), P^\nu] = d^\nu\psi(x) + \frac{2g^2}{3M^2}\left(1 - \frac{2g^2}{3M^2}(\nabla\phi)^2\right)^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi(x) \tag{3.5}$$

then from (1.1), consistency will demand

$$\partial^\nu\psi(x) = -\frac{2g^2}{3M^2}\left(1 - \frac{2g^2}{3M^2}(\nabla\phi)^2\right)^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\phi)(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi(x). \tag{3.6}$$

Similarly, we can calculate

$$-i[\psi_k^{3/2}(x), P^\nu] = d^\nu\psi_k^{3/2}(x) + \tfrac{2}{3}igP_{kl}\,d_l[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi(x); \tag{3.7}$$

again from the consistency requirement

$$\partial^\nu\psi_k^{3/2}(x) = -\tfrac{2}{3}igP_{kl}\,d_l[M^2 - \tfrac{2}{3}g^2(\nabla\phi)^2]^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\partial^\nu\phi)\psi(x). \tag{3.8}$$

Then, from the definition

$$\partial^\nu \chi = \frac{\partial \chi}{\partial \psi_k^{3/2}} \partial^\nu \psi_k^{3/2} + \frac{\partial \chi}{\partial \psi} \partial^\nu \psi + \frac{\partial \chi}{\partial \phi} \partial^\nu \phi. \tag{3.9}$$

Using the above results and the primary constraint (2.4), we get

$$\partial^\nu \chi = (\tfrac{2}{3}\mathrm{i}\boldsymbol{\gamma} \cdot \mathbf{d} + M)^{-1}[-\mathrm{i}\, \mathbf{d} \cdot \partial^\nu \boldsymbol{\psi}^{3/2} + g(\boldsymbol{\gamma} \cdot \boldsymbol{\partial}\phi)\partial^\nu \psi + g\gamma_k \psi\, \partial_k\, \partial^\nu \phi]. \tag{3.10}$$

Now we can evaluate $-\mathrm{i}[\chi(x), P^\nu]$ from the primary constraint (2.4), namely

$$-\mathrm{i}[\chi(x), P^\nu] = (M + \tfrac{2}{3}\mathrm{i}\boldsymbol{\gamma} \cdot \mathbf{d})^{-1}\{-[\mathrm{i}\, \mathbf{d} \cdot \boldsymbol{\psi}^{3/2}, P^\nu] + g(\boldsymbol{\gamma} \cdot \boldsymbol{\partial}\phi)[\psi, P^\nu]\}. \tag{3.11}$$

This becomes,

$$-\mathrm{i}[\chi(x), P^\nu] = \mathrm{d}^\nu \chi - (\tfrac{2}{3}\mathrm{i}\boldsymbol{\gamma} \cdot \mathbf{d} + M)^{-1}[-\mathrm{i}\, \mathbf{d} \cdot \partial^\nu \boldsymbol{\psi}^{3/2} + g(\boldsymbol{\gamma} \cdot \boldsymbol{\partial}\phi)\, \partial^\nu \psi + g\gamma_k \psi\, \partial_k \partial^\nu \phi]. \tag{3.12}$$

In other words, from (3.10)

$$\mathrm{d}^\nu \chi(x) = -\mathrm{i}[\chi(x), P^\nu] + \partial^\nu \chi(x) \tag{3.13}$$

which is compatible with the Heisenberg equations of motion (1.1).

This allows us to conclude that the quantised spin-$\frac{3}{2}$ field in interaction with a Dirac field and an external scalar field in the manner considered above, does obey the Heisenberg equations of motion, and from the expression (3.2) we find the latter are the same as the Lagrangian equations of motion.

Since the quantisation of the interacting field can be carried out only in the weak-field limit, $(2g^2/3M^2)(\nabla\phi)^2 < 1$, the above discussion of the Heisenberg equations of motion is also valid under this limit. In the present case with ϕ external, the choice of canonical variables seems arbitrary (Nagpal 1974) when quantisation is carried out using Schwinger's action principle approach. The arbitrariness can be removed in the fully quantised theory or resorting to the Yang–Feldman technique, and a unique set of canonical variables emerges. The question of the validity of the Heisenberg equations of motion for a spin-$\frac{3}{2}$ field operator, in a fully quantised interaction, will be studied elsewhere.

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